Indian Statistical Institute Bangalore
Statistics and Mathematics Unit
M. Math II Year
Course: Several Complex Variables
Midterm Exam, Date: 19 February, 2025

Time: 3 hours Maximum marks: 50

All questions are compulsory.

NOTATIONS: (1) $\mathbb{R}$: set of all real numbers, (2) $\mathbb{C}$: set of all complex numbers, (3) $\mathbb{C}^n$: the n-dimensional complex number space, (4) $\mathbb{R}^n$: the n-dimensional real number space, (5) $[a,b]$: closed and bounded interval in $\mathbb{R}$, (6) For a given $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ and $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ with $r_i > 0$ for all $1 \leq i \leq n$,

$$\mathbb{D}^n(\mathbf{a}, \mathbf{r}) := \{\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_i - a_i| < r_i \text{ for all } 1 \leq i \leq n\}.$$

For $n = 1, a \in \mathbb{C}$ and a positive $r \in \mathbb{R}$, we will write $\mathbb{D}(a, r)$, instead of $\mathbb{D}^1(\mathbf{a}, \mathbf{r})$. (7) $\|\cdot\|_K$ denotes the supremum over a given set K, (8) $A \times B$: cartesian product of two sets A and B, (9) $\exp(z)$: the exponential function e^z for $z \in \mathbb{C}$.

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1. (a) Prove that the function

$$I(h) = \exp\left(\int_0^1 \log \frac{h^2 + x^2}{x^2} dx\right)$$

from $[0,1]$ to $[1, \infty)$ is well-defined, strictly increasing and continuous, with $I(0) = 1$. **(5 marks)**

(b) Consider a compact subset K of $\mathbb{C}$, a closed subset B of K and $a \in K$. Moreover, assume that there is $\rho > 0$ such that the set $\{t \in [0, \rho] : \mathbb{T}(a,t) \cap B \neq \emptyset\}$ has positive Lebesgue measure m ($\mathbb{T}(a,t)$ is the circle with radius t, centered at a). Prove that for every polynomial $P : \mathbb{C} \to \mathbb{C}$ and every real number $r > 0$,

$$\|P\|_{\mathbb{D}(a,r)} \leq \|P\|_B I(\alpha)^{\deg(P)}$$

where $\deg(P)$ denotes the degree of polynomial P, $I(\alpha)$ is as defined in part (a), and

$$\alpha = \sqrt{\frac{\rho + r - m}{m}}.$$

(12 marks)

(c) Given an open set $D \subset \mathbb{C}^n$ ($n \geq 2$), let $f : D \to \mathbb{C}$ be separately holomorphic and locally bounded on D. Show that f is continuous on D. **(4 marks)**

2. (a) Let the complex-valued function $F(z, s)$ be defined for $(z, s) \in \Omega \times [0, 1]$, where Ω is an open set in $\mathbb{C}$. Also suppose that $F(z, s)$ is continuous in $\Omega \times [0, 1]$ and is holomorphic in Ω for each $s \in [0, 1]$. Show that the function f defined on Ω by

$$f(z) = \int_0^1 F(z, s) ds$$

is holomorphic. **(4 marks)**

(b) Using part (a), prove that if f is a complex-valued holomorphic function defined on the domain

$$\left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, \frac{1}{2} < |z_2| < 1\right\} \bigcup \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_2| < 1, \frac{1}{2} < |z_1| < 1\right\},$$

then f has an analytic continuation to the unit bidisc $\mathbb{D}^2(\mathbf{0}, \mathbf{1})$, where $\mathbf{0} = (0, 0) \in \mathbb{C}^2$, $\mathbf{1} = (1, 1) \in \mathbb{R}^2$. **(7 marks)**

3. (a) Let D be an open connected set in $\mathbb{C}^n\,(n \geq 2)$ and f be a non-constant, complex-valued holomorphic function defined in D. Then show that $f(\Omega)$ is open for any open set $\Omega \subset D$. **(4 marks)**

   (b) Suppose $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}^2$ is a holomorphic map, defined by

$$f(z) = \big(z(z-1), z^2(z-1)\big)\,.$$

   Determine, with justification, whether f is open or not (we say f is open if $f(\Omega)$ is open for all open subsets Ω of $\mathbb{C} \setminus \{1\}$). **(4 marks)**

4. (a) Find the value of the integral

$$\int_{|z_1|=1} \int_{|z_2|=1} \frac{\exp(z_1 z_2)}{|(z_1 - 2)(z_2 - 2)|^2} dz_2 dz_1$$

(5 marks)

   (b) Determine, with justification, the maximum of the following sets:

$$\left\{|2z_1 + 10z_2 + 11z_3| : (z_1, z_2, z_3) \in \mathbb{C}^3 \text{ and } |z_1|^2 + |z_2|^2 + |z_3|^2 \leq 1\right\}$$

   and

$$\left\{\left|z_1^2 z_2 z_3 - z_2^2 z_1 z_3 + z_3^2 z_1 z_2\right| : (z_1, z_2, z_3) \in \mathbb{C}^3 \text{ and } |z_1| \leq 1, |z_2| \leq 1, |z_3| \leq 1\right\}.$$

(2.5 marks+2.5 marks)

Best wishes!